Higher Math via BG

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1 The classical story

[**?**] For a topological group G then $EG \to BG$ is a fiber bundle with structure group G such that for any fiber bundle, over a finite dimensional CW complex, with structure group G, $T \to X$ we have that $T \to X$ is isomorphic to the pull back

$$\begin{tikzcd} T \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EG \arrow[d] \\ X \arrow[r] & BG \end{tikzcd}$$

The spaces BG exists for all topological groups and the map $X \to BG$ is unique up to homotopy.

Lemma ([Mil56])**.** *BG exists for a topological group G.*

> ***Proof.*** What we describe here is Milnors elegant construction. We call a G principle bundle n-universal if it classifies G bundles whose total spaces are $(n-1)$-connected.
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> Milnor claims that the total space for the n-universal G bundle is given by
>
> $$E_n = G * \cdots * G$$
>
> that is the $(n+1)$-fold join of G with itself (recall that the join is attaching a line between any two points).
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> This total space has a natural action of G by acting on the end points of the line. For instance if we consider a line in $G * G$ joining g_1, g_2 then by acting by h we would get the line joining g_1h, g_2h. Thus a point in $G * G$ which is a point along one of these lines, say indexed by 'time' t will be sent to the point on the line g_1h, g_2h 'at the same time'.
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> We can take the quotient of E_n by this action to obtain the base space B_n, and the projection gives the bundle map. Milnor then remarks that one can obtain the proper universal bundle by setting $n = \infty$. As discussed here one must be delicate with the topology imposed on the join, and

> in particular in the case of $n = \infty$. One is tempted to define E_∞ as the colimit over the E_n, and this is true, provided that the group acts freely and continuously on the colimit. In particular it is sufficient for G to be locally compact for the following to be true
>
> $$EG = \mathrm{colim}_n G^{*n}.$$
>
> and since quotients commute with colimits we also have that
>
> $$BG = EG/G.$$

The first moral is that taking repeated joins increases the connectedness of the space. The second moral is to examine exactly what the quotient is doing homotopically. First consider the two fold join $G * G$. The joins connects all group elements by a line, for a line $g \to h$ the action of g^{-1} gives a line $e = gg^{-1} \to hg^{-1}$ and these two lines are identitfied under the quotient. In particular the end points of all lines will be identified with e!

The case of a discrete group is easy to see more in a “geometric” way. The join will be just a complete bipartite graph on the elements of G, that is it will have two copies of the elements of G and all the elements of one copy connect to all the elements of the other. Now when we take the quotient we see that the two “sides” of the graph collapse and we join “parrallel” lines (not literally). Homotopically this is two points with a bunch of lines between them, or equivalently one point with a bunch of loops. Now up to homotopy the only thing that is important is *the number of the lines*. Purely combinatorially though we started with $|G|^2$ lines and we identified $|G|$ of them leaving us with $|G|$ left over. Hence we have a bouque of $|G|$ circles. Thus the *one skeleton* of BG is identified with the elements of G!

The next join can be analysed similarly and seen to be introducing a two cell to this structure for each relation $gh = f$. The boundary of the two cell will trace g, then h and then f^{-1} which enforces the relation $ghf^{-1} = e$. This is then continued for higher cells and higher relations. Note that this is only up to homotopy, its not litterally what is identified in the join construction.

Lemma. *$\pi_n BG \cong \pi_{n-1} G$.*

> ***Proof.*** Follows immediately from the contractability of the total space and the LES of a fibration.

Remark. Milgram remarks ”The existence of BG follows easily from Brown representation theorem”.

Remark. [Mil56, Thm 5.2] is interesting as it establishes an equivalence

$$\text{countable CW groups up to equivalence} \xrightarrow{B} \text{countable connected CW complexes up to homotopy}$$

where two topological groups are declared equivalent iff they contain an isomorphic subgroup and both spaces are homotopy equivalent to that subgroup. The interesting part is the surjectivity of this map.

2 Up one categorical level

For a group G there is a natural one category BG. This category has one element and the Hom set of this element is given by G. Clearly this is a category as one can “compose” morphisms perfectly well and there is an identity.

If one thinks of a discrete group then this category has a natural interpretation of a topological graph, the objects being the verticies (theres only one) and the morphisms being edges (all loops). It is clear that as a topological graph this is again the one skeleton of the construction above, as there is a single point with $|G|$ loops.

3 To infinity

What is the nerve of this category? Well its zero simplicies are again just a single point, and one simplicies are just the group elements. Then two simplicies are compositions of one morphisms, that is multiplications of group elements etc etc. The face maps for such a word in the group is given by composition which means in this case essentially multiplying the group elements.

$$\mathrm{face}_3^1(abc) = (ab)c$$

that is evaluate the multiplication ab and then multiply c. The degeneracy maps are the insertion of the identity of the group.

This produces a simplicial set whose n simplicies are length n words in G and whose face maps are evaluating the multiplication. We now want to go through the explicit geometric realisation of this simplicial set. This of course can be defined as a certain colimit but we want to slow down. If we denote X_n the collection of n-simplicies of $N(BG)$, that is the collection of length n words in G, with the discrete topology then from [**?**] we have that

$$|N(BG)| = \coprod_n X_n \times \Delta^n / \sim$$

where we identify $(x, D_i(p)) \sim (d_i(x), p)$ and $(x, S_i(p)) \sim (s_i(x), p)$ where d, D are the FACE maps of the simplicial set and standard simplicial space respectively and s, S are the DEGENERACIES. Note that while in the nerve the face maps send you *down* a degree the face maps are on the level of spaces the inclusion of the face into a higher simplex and hence send you *up* a degree, the functor is contravariant and that is why the direction swaps.

Lets think of what this means. The space X_n is discrete so the product essentially just gives us a Δ^n for each n-simplex. The first relation then identifies the n-simplex $d_i(x)$ and glues it as the i-th face to the $n+1$-simplex (associated to) x. If it happens that $d_j(y) = d_i(x)$ then we see that the two simplicies x, y are glued along that face. The second relation is there to kill off degenerate simplicies. If $y = s_i(x)$ is a degenerate simplex, then we just collapse it to x.

For BG the first relation says that for a word ab if it multiplies to c, that is $ab = c$, or more accurately $abc^{-1} = e$, then we should glue the ab cell along a face to the lower degree cell c^{-1}. In general we see that if two elements multiply to a third then the cell of the two element word is glued to the lower cell of the single word, and this will also take place for longer words and subwords. We also have however another bracketing of abc^{-1} to bc^{-1} ($bc^{-1} = a$) and so we have to glue the a loop to bc^{-1}. We want to compare these two gluings and that should be done with a higher cell. On the other hand the word ae is clearly the degeneration of a and hence should be glued to a. We can see that we recover the homotopy type of the Milnor construction.

Remark. It pisses me off that people are using d for face when there is also the degeneracy maps.

Remark. The nerve of a category is "trivial" as an infinity category; an often said phrase. We can see here that simplicial set given by the nerve of a category is far from trivial and the geometric realisation is far from a contractable space. What is meant is that the *higher morphisms* are contractable or strictly the identity. Thus we must be careful to not conflate the n-simplicies with the n-morphisms in an infinity category! This is the non-trivial part of constructing the mapping spaces of an infinity category.

4 Homotopy Quotients: Introducing stacks

A quotient for topological spaces is a colimit (more specifically the coequaliser). If we have a space X then a quotient by some relation $R \subseteq X \times X$ is a colimit of the diagram

$$\begin{tikzcd} R \arrow[r, hook] & X \times X \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \end{tikzcd}$$

Example. *Let $Y \subseteq X$ and consider the quotient X/Y where we collapse Y to a point. As an equivalence relation we want to identify all the points in Y so $R = \{y_0\} \times Y \subseteq X \times X$ for some point $y_0 \in Y$. The coequaliser is then the space Q with the map $f : X \to Q$ that coequalises the projections, that is $f \circ \pi_1 = f \circ \pi_2$. Explicitly that is $f(y_0) = f(y)$ for every $y \in Y$. Now consider the pushout*

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d] & X \arrow[d] \\ * \arrow[r] & * \sqcup_Y X \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

Which comes with a map from X, and clearly satisfies the universal property.

Example. *Let $G \curvearrowright X$ and both be nice spaces, we want to look at the quotient of X under this group action, that is the collection of orbits. Then the equivalence relation R is given by $x \sim y$ iff $\exists g \in G$ such that $gx = y$. This is the same as $\{(x, gx) : g \in G, x \in X\} \cong X \times G, (x, gx) \leftarrowtail (x, g)$. We need a map $f : X \to Q$ that coequalises the group action, explicitly $f(x) = f(gx)$. Clearly the 'orbit space' $X/G = \{Gx : x \in X\}$ and the natural projection satisfy this property.*

Now we have this formulation we can ask the simple question: what is the *homotopy* colimit of the diagram? The very classical construction of a homotopy colimit is as follows [Bou87]. We no longer requre that the diagram be strictly coequalized, that is $f(x) = f(gx)$ but only that there is a path between them, likewise we glue in a two cell for relations $f(gx) = f(hgx)$ etc. That is applying [Shu09, §7] the bar construction to the diagram. We have seen that the Bar construction applied to $N(BG)$ returns the classical BG and similarly there is a Bar construction that can be applied to a simplicial EG, which is basically just n-simplicies as G^{*n}. Geometric realisation commutes with quotients and so we can see that the quotient of the Bar construction is the Bar construction of the quotients. Thus the Bar construction which by definiton gives us the homotopy colimit can be seen to also produce $(EG \times X)/G$ which is also given as the Bar construction of some diagram. More or less. Flesh this out a bit more. Really define the Bar construction and do it. This is known as the Borel construction and is therefore a model for the "homotopy quotient" of X by G.

Now consider the action of a group on a point $*$. clearly there is only one such action and its trivial. What is the homotopy quotient? We perform the Borel construction $(* \times EG)/G \simeq EG/G \simeq BG$! Thus we can see that the *homotopy* of a point can be a *non-contractable space*! The point is that we replace our *actual point* with something contractable. Then when we take the quotient we get something (potentially) highly non-trivial. *If I was an idiot I might call this a stacky quotient...*

If G is discrete then we could categorify this picture to see group actions on a space as a functor $BG \to$ Top, it picks out a space and sends the group elements to endormorphisms of that space. Im not sure what you get out of this perspective.

5 Coherent Nerves: Into the Void

So we have seen that for a general group we can construct a classifying space. We also saw that *for a discrete group* this Milnor construction was homotopic to the so called Bar construction applied to the nerve of the one category associated to the group; one element and an automorphism for each group element. We have also seen that this construction would not returne BG when the group is not discrete as the category does not retain any of the topology.

Now we will go infinity categorical. From G as a group we can construct *an infinity category* (in fact an infinity groupoid). The idea is exactly as before, we have a category with a single object and the collection of maps is G, however now we have a topology on G. Categories enriched in Top form a model of infinity categories, which we can map to the quasi-category model by first taking Sing on all the hom sets (changing to a simplicially enriched category) and then applying the coherent nerve [**?**, §1 somewhere]. This process alows us to now perform the one categorical computations we had for discrete groups but with any group, namely we can think of (any) group action now as an infinity functor into the infinity category of spaces

$$BG \to \mathcal{S}.$$

Whats the point of this all. One is that the categorical and homotopical reformulations make all the group theory things more uniform, not requiring sort of particular definitions everytime we add a group to a structure. The other point is that looking at the category (when BG is a one category) or space

$$\mathrm{Func}(BG, \mathcal{S})$$

we see that it is category of presheaves and hence has many nice properties.

References

[Bou87] Aldridge K. Bousfield. *Homotopy Limits, Completions and Localizations.* Number v.304 in Lecture Notes in Mathematics Ser. Springer Berlin / Heidelberg, Berlin, Heidelberg, 1987.

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[Shu09] Michael Shulman. Homotopy limits and colimits and enriched homotopy theory, July 2009. arXiv:math/0610194.